Problem 1. Let R be the subring of $\mathbb{C}[x]$ consisting of all polynomials $f(x) \in \mathbb{C}[x]$ such that $f'(0) = 0$; i.e, the derivative of f at 0 is 0. Is R a finitely generated ring over $\mathbb{C}$? If yes, find an isomorphism from R to some quotient of a polynomial ring over $\mathbb{C}$ (with finitely many indeterminates). If no, justify your answer.

Problem 2. In this problem K is either the field of real numbers or the field of complex numbers. Let $\mathcal{N}$ denote the set of nilpotents in $M_{n\times n}(K)$, the set of $n \times n$ matrices with entries in K, and $\mathcal{U}$ the set of unipotents – i.e., matrices A such that $A - 1_{n\times n}$ is nilpotent. Show that the exponential map

$$\exp\,:\, M_{n\times n}(K) \longrightarrow M_{n\times n}(K)\ ,\quad \exp A \;=\; \textstyle\sum_{n=0}^{\infty} \frac{A^n}{n!}$$

maps $\mathcal{N}$ one-to-one onto $\mathcal{U}$. Can you describe the inverse of this map?

2b) What can you say about the exponential map from $M_{n\times n}(K)$ to itself? What is its image? Is it invertible in any sense?

Problem 3. Let p be a prime, $\mathbb{F}_p$ the prime field of p elements, and ζ_p a primitive p-th root of unity in $\mathbb{C}$. For a positive integer d, define the algebraic integer

$$G_d = \sum_{x\in\mathbb{F}_p} \zeta_p^{x^d}.$$

Prove that the degree over $\mathbb{Q}$ of the algebraic integer G_d is equal to $(d, p-1)$.

Problem 1. Let $d \in \mathbb{N}$ and $\zeta := e^{\frac{2\pi\sqrt{-1}}{d}}$ be a d-th primitive root of unity. Let A be the $(d-1) \times (d-1)$ matrix whose (i,j) entry is $A_{ij} = \zeta^{ij} - \zeta^{(i-1)j}$. Show that $\det(A)^2 \in \mathbb{Z}$. For $d \equiv 0, 3 \pmod 4$ show that $\det(A) \notin \mathbb{Z}$.

Problem 2. Let k be any field of characteristic not equal to 2. Let M be an $n \times n$ orthogonal matrix over k; that is, the coefficients of M are in k, and $MM^t = I_n$. Assume that n is odd. Prove that M has an eigenvalue equal to $\det(M)$.

Problem 3. Let $G \subset GL_n(\mathbb{Z})$ a finite subgroup. Prove that there is a constant c_n depending only on n such that the order of G satisfies: $|G| \leq c_n$.